Econ 413R: Computational Economics
Spring Term 2013

Dynamic Stochastic General Equilibrium Modeling

1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are models of general equilibrium where the economy's equilibrium changes over time due to stochastic shocks.

1.1 General Equilibrium

A general equilibrium is a situation for the economy as a whole where all markets are in equilibrium, with supply equaling demand at the prevailing prices. A competitive equilibrium is a special case of general equilibrium where we satisfy certain conditions.

Under the right set of conditions a competitive equilibrium is identical to the solution to a central planner's problem. Many early DSGE models were set up and solved this way. However, the necessary conditions for this equivalence to hold often do not apply in current state-of-the-art macro models.

For example, models where firms are monopolistically competitive will not generate this equivalence. Solving a social planner's problem for these models will yield the socially optimal outcome, but this is not the outcome that will be generated by a decentralized set of markets and optimizing agents. For that reason, most current DSGE models specify and solve a competitive equilibrium.

Definition 1. A *competitive equilibrium* is a set of allocations, $\{x_i\}_{i=1}^{I}$, and prices, $\{p_i\}_{i=1}^{I}$, for each factor of production and consumable such that:

  i. households optimize utility,

 ii. firms optimize profits,

iii. the government meets its budget constraints, and

iv. all factor and goods markets clear

1.2 Walras' Law

Walras' Law is a useful tool in general equilibrium modeling. It states that if an economy has N markets and $N-1$ of those markets are in equilibrium, then the remaining market must also be in equilibrium. This is useful because it means we need only incorporate the market clearing conditions from $N-1$ markets in our model, and we can ignore one market by invoking Walras' Law. For example, if our model has a goods market, a labor market, and a market for capital, we could ignore the goods market clearing condition. We would need only to incorporate the two factor market clearing conditions.

1.3 DGE and DSGE models

A general equilibrium model becomes dynamic when we incorporate time. Specifically, conditions of the economy in one moment are determined in part by the past and will influence the future in some way.

A dynamic general equilibrium (DGE) model incorporates this time element using either a continuous or discrete formulation of time. Continuous time modeling is widely used in the economic growth literature, while the literature on economic fluctuations almost always uses a discrete time setup. Each setup has its advantages. Continuous time modeling allows for the use of tools from the analysis of differential equations to solve our models. These tools are well-understood and widely used in many contexts. Discrete-time modeling requires the use of difference equations which are very similar, but are less widely used. Discrete-time modeling is very convenient when the goal is to numerically solve and/or simulate a model on a computer. Ultimately, computers must discretize any problem, so it is often very useful to have a model that is already described in these terms. These notes present models using a discrete-time setup precisely because these models are intended to be solved and simulated on computers.

Dynamic *stochastic* general equilibrium (DSGE) models are a special class of DGE models. DSGE models incorporate at least one stochastic variable that changes over time. Often this is a shock to productivity, but many large-scale DSGE models also incorporate a large number of other types of shocks. Since the shocks have a random component, they are modeled as stochastic processes, which are often referred to as "laws of motion." Often these laws of motion are simple stochastic processes like an AR(1) or a random walk.

Strictly speaking the "shocks" are the purely random innovations to the stochahstic process – the ϵ^z in (2.16) below. However, we also often use the term "shock" to refer to the variable upon which these innovations impact – the z in the same equation.

2 A General DSGE Model

A typical DSGE model specifies a household's problem, a firm's problem, market clearing conditions, and laws of motion for the stochastic shocks to the system. This section presents a fairly general framework that characterizes most DSGE models. In sections 3 and 4 below we work through a concrete example for two relatively simple DSGE models.

2.1 Household's Problem

There is a unit measure of households, each assumed to live forever and to maximize expected lifetime utility subject to a series of period-by-period budget constraints.

$$\max_{\{x_{it}, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E_t\{u(\{x_{it}\}_{i=1}^{I})\}$$

$$\text{s.t.} \sum_{i=1}^{I} p_{it}x_{it} + (1 + r_t - \delta)k_t - k_{t+1} = 0 \quad \forall t \tag{2.1}$$

where the x_{it} is the net demand for good i by the household (consumption being a positive number) at price p_{it} in period t, k_t is savings in the form of capital goods from period $t-1$ that pays interest r_t in period t, $u(.)$ is a

within-period utility function, β is a number between zero and one that discounts future utility, and δ is the rate of capital depreciation. The prices, p_{it}, and capital rental rate, r_t, are potentially stochastic, making utility stochastic as indicated by the expectations operator, E_t, in (2.1).

One way to formally incorporate this uncertainty would be to consider all possible cases for the realization of these prices. If the number of cases was countable, for example, we could index each possible realization or "state of nature" with an s subscript and rewrite the problem as follows.

$$\max_{\{x_{ist}, k_{s,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s} \pi_{st} u(\{x_{it}\}_{i=1}^{I})$$

$$\text{s.t.} \sum_{i=1}^{I} p_{ist} x_{ist} + (1 + r_{st} - \delta) k_{st} - k_{s,t+1} = 0 \quad \forall s, t \qquad (2.2)$$

where π_{st} is the probability of state of nature s occurring in time period t.

This formulation quickly becomes very complicated as, in general, we need to define the state of nature as the history of stochastic shocks and not just the current values for a set of stochastic variables. As we have written the problem above, the household solves the utility maximization problem for all eternity over all states of nature, just once at the beginning of the economy. Once these consumption decisions are made, the household simply follows this preset contingency plan each period depending on which state of nature is realized.

A more promising approach is to imagine optimization as occurring sequentially. This approach is an application of dynamic programming. We need to assume that the household's problem meets certain regularity con-

ditions before we can jump from (2.2) to dynamic programming. These conditions amount to assuming that the household solves the same kind of problem each period. For infinitely-lived agents this will be true. It would not be true for agents with finite lifetimes as we have in overlapping generations models, however, and the dynamic programming tools we use below would not be applicable.

A dynamic program is expressed by a Bellman equation:

$$V(k_t; \theta_t) = \max_{x_{it}, k_{t+1}} u(\{x_{it}\}) + \beta E_t\{V(k_{t+1}, \theta_{t+1})\}$$

$$\text{s.t. } \sum_{i=1}^{I} p_{it} x_{it} + (1 + r_t - \delta)k_t - k_{t+1} = 0 \qquad (2.3)$$

$V(.,.)$ is referred to as the value function. In our context this corresponds to the expected value of all future lifetime utility at period t. θ_t is a vector of exogenously determined variables that may follow some stochastic process.

There are many ways to solve a dynamic programming problem. We will discuss this in much greater detail later.

One set of conditions that should be obvious are the first-order conditions for the maximization problem inside the Bellman equation. λ_t below is the Lagrange multiplier for the period t budget constraint.

$$u_{x_i}(\{x_{it}\}_{i=1}^{I}) + \lambda_t p_{it} = 0 \quad \forall i \qquad (2.4)$$

$$-\lambda_t + \beta E_t\{V_k(k_{t+1}, \theta_{t+1})\} = 0 \qquad (2.5)$$

The condition on each λ_t recovers the budget constraint.

There is also an envelope condition from the envelope theorem. The envelope theorem states that if $f(x, p)$ attains its maximum with respect to x, $f^*(p) = \max_x f(x, p)$, at $x^*(p)$, then $\frac{d\, f^*(p)}{dp} = \left.\frac{\partial f(x,p)}{\partial p}\right|_{x=x^*(p)}$. In our case this condition this implies:

$$V_k(k_t; \theta_t) = \lambda_t(1 + r_t - \delta) \tag{2.6}$$

Combining (2.5) and (2.6) gives us what is known as an Euler equation.

$$\lambda_t = \beta E_t\{\lambda_{t+1}(1 + r_{t+1} - \delta)\} \tag{2.7}$$

Substituting in (2.4) for λ_t puts this in a more common form expressed using marginal utilities over time.

$$\frac{u_{x_i}(\{x_{it}\}_{i=1}^I)}{p_{it}} = \beta E_t\left\{\frac{u_{x_i}(\{x_{i,t+1}\}_{i=1}^I)}{p_{i,t+1}}(1 + r_{t+1} - \delta)\right\} \quad \forall i \tag{2.8}$$

2.2 Firm's Problem

A unit measure of firms arise spontaneously each period. Each firm rents capital and buys other factors from households. Free entry and competitive markets drive their profits to zero. The firm's problem is not dynamic as the firm exists only for one period. Shocks to the production function occur each period.

$$\max_{K_t, X_{it}} \Pi_t = \sum_{i=1}^I P_{it}X_{it} - R_tK_t \text{ s.t. } F(\{X_{it}\}, K_t, z_t) = 0$$

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where P_{it} is the price of good i in period t faced by the firm, $F(.)$ is a multivariate production function over the X_{it}'s with outputs being positive and inputs being negative, K_t is the capital hired, R_t is the rental rate paid for capital, and z_t is an aggregate technology shock that is common to all firms. The Lagrange multiplier for the production constraint is denoted Λ_t.

Maximization yields the following first-order conditions:

$$P_{it} + \Lambda_t F_{Xi}(\{X_{it}\}_{i=1}^I, K_t, z_t) = 0 \quad \forall i \tag{2.9}$$

$$-R_t + \Lambda_t F_K(\{X_{it}\}_{i=1}^I, K_t, z_t) = 0 \tag{2.10}$$

$$F(\{X_{it}\}_{i=1}^I, K_t, z_t) = 0 \tag{2.11}$$

2.3 Adding-Up and Market Clearing

Market clearing conditions for period t are given by (2.12) and (2.13).

$$x_{it} = X_{it} \quad \forall i \tag{2.12}$$

$$k_t = K_t \tag{2.13}$$

We also must impose equivalences between the prices faced by households and those faced by firms to prevent infinite arbitrage opportunities.

$$p_{it} = P_{it} \quad \forall i \tag{2.14}$$

$$r_t = R_t \tag{2.15}$$

2.4 Exogenous Laws of Motion

Finally, we need to specify a stochastic process for the technology shocks:

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.}(0, \sigma_z^2) \tag{2.16}$$

2.5 The Equilibrium

Recall that a competitive general equilibrium is defined by the conditions that 1) the households are maximizing utility, 2) the firms are maximizing profits, and 3) all markets clear (note there is no government in this model). Hence, in this general model the competitive general equilibrium is given by (2.3) and (2.8) – (2.16). The variables in the system are: $\{x_{it}, p_{it}, X_{it}, P_{it}\}_{i=1}^I$, k_t, K_t, r_t, R_t, Λ_t and z_t.

Note the system can be simplified considerably by imposing (2.12) – (2.15) on (2.9) – (2.11) to eliminate the variables denoted with capital letters. Also, (2.10) can be used to eliminate Λ_t. This leaves us with a system of $2I + 3$ equations in $2I + 3$ variables.

It is often helpful to group the variables into categories. First we can list variables that are exogenous to the model. We will use the notation from Uhlig (1999) and call these Z_t. In this case, it consists only of z_t. All other variables are endogenous to the model. There is a generally non-unique set of these variables that collectively with Z_t define the state of the economy. This set of endogenous state variables are denoted X_t. In our case a straightforward choice is to use k_t. Looking carefully at the equations of our model, we find that knowing k_{t-1} along with z_t gives us everything we need to solve for all the other variables in the current period, including k_t.

The set of endogenous non-state variables (also called "jump" variables) is denoted Y_t. To summarize we have:

$$X_t = \{k_{t-1}\}$$
$$Y_t = \left\{ \{x_{it}, p_{it}\}_{i=1}^I, r_t \right\} \tag{2.17}$$
$$Z_t = \{z_t\}$$

Our goal is to find a policy function, Φ, that maps the values of the current state into the values for the endogenous state variables next period.

$$X_{t+1} = \Phi(X_t, Z_t) \tag{2.18}$$

Before we discuss ways to do this, let's first look a less general model.

3   Our Baseline Model

In this section we will consider a variant of the model in Hansen (1985). The model is a special case of the general one presented in the previous section with the exception of the addition of a government budget constraint.

3.1   Household's Problem

Households in this model hold capital (k_t) and an endowment of labor which is normalized by a choice of units to one. They earn a wage rate (w_t) payable on the portion of this labor endowment they choose to supply to the market, and generate utility with the remaining labor, which we can think of as

leisure. They also earn a rental rate (r_t) on their capital, but lose a fraction (δ) to depreciation. There is also a government in our version of the model, which is missing from Hansen's specification. The government taxes household income at a constant marginal rate (τ) and refunds the proceeds lump-sum to the households in the form of a transfer (T_t). From this net income, households choose a consumption amount (c_t) and an amount of capital to carry over to the next period (k_{t+1}).

The dynamic program for the households is

$$V(k_t; \theta_t) = \max_{\ell_t, k_{t+1}} u(c_t, \ell_t) + \beta E_t\{V(k_{t+1}, \theta_{t+1})\} \tag{3.1}$$

$$\text{s.t. } (1 - \tau)\left[w_t\ell_t + (r_t - \delta)k_t\right] + k_t + T_t = c_t + k_{t+1} \tag{3.2}$$

We can dispense with the Lagrangian if we rewrite (3.2) as

$$c_t = (1 - \tau)\left[w_t\ell_t + (r_t - \delta)k_t\right] + k_t + T_t - k_{t+1} \tag{3.3}$$

and substitute it into the utility function of (3.1).

The first order conditons from the problem are:

$$u_\ell(c_t, \ell_t) + u_c(c_t, \ell_t)w_t(1 - \tau) = 0 \tag{3.4}$$

$$- u_c(c_t, \ell_t) + \beta E_t\{V_k(k_{t+1}, \theta_{t+1})\} = 0 \tag{3.5}$$

The envelope condition is :

$$V_k(k_t; \theta_t) = u_c(c_t, \ell_t)[(r_t - \delta)(1 - \tau) + 1] \tag{3.6}$$

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Combining (3.5) and (3.6) and rearranging terms gives us the intertemporal Euler equation.

$$u_c(c_t, \ell_t) = \beta E_t \left\{ u_c(c_{t+1}, \ell_{t+1})[(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \tag{3.7}$$

Rewriting (3.4) gives the a consumption-leisure Euler equation.

$$- u_\ell(c_t, \ell_t) = u_c(c_t, \ell_t)w_t(1 - \tau) \tag{3.8}$$

Note that work generates disutility via lost leisure, so the left-hand side is the utility from an additional unit of leisure.

3.2 Firm's Problem

As in section 2.2, a unit measure of firms arises spontaneously each period. Each firm rents capital and labor services from households. The objective is to maximize profits as shown.

$$\max_{K_t, L_t} \Pi_t = f(K_t, L_t, z_t) - W_t L_t - R_t K_t$$

where K_t and L_t are capital and labor hired, R_t and W_t are the factor prices, and $f(.)$ is the production fuction.

It yields the following first-order conditions:

$$R_t = f_K(K_t, L_t, z_t) \tag{3.9}$$

$$W_t = f_L(K_t, L_t, z_t) \tag{3.10}$$

3.3 Government

The government collects distortionary taxes and refunds these to the households lump-sum:

$$\tau\left[w_t\ell_t + (r_t - \delta)k_t\right] = T_t \qquad (3.11)$$

3.4 Adding-Up and Market Clearing

Market clearing is:

$$\ell_t = L_t \qquad (3.12)$$

$$k_t = K_t \qquad (3.13)$$

Price equivalences are:

$$w_t = W_t \qquad (3.14)$$

$$r_t = R_t \qquad (3.15)$$

3.5 Exogenous Laws of Motion

The stochastic process for the technology is the same as in section 2.4.

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.}(0, \sigma_z^2) \qquad (3.16)$$

3.6 The Equilibrium

The dynamic equilibrium for the model is defined by (3.3) and (3.7) – (3.16), a system of eleven equations in eleven unknowns. We can simplify this, however, by using (3.14) and (3.15) as definitions to eliminate the variables W_t and R_t. Similarly, (3.12) and (3.13) eliminate L_t and K_t. This leaves

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us with seven equations in seven unknowns, $\{c_t, k_t, \ell_t, w_t, r_t, T_t \ \& \ z_t\}$. The equations are:

$$c_t = (1 - \tau)\left[w_t\ell_t + (r_t - \delta)k_t\right] + k_t + T_t - k_{t+1} \tag{3.17}$$

$$u_c(c_t, \ell_t) = \beta E_t\left\{u_c(c_{t+1}, \ell_{t+1})[(r_{t+1} - \delta)(1 - \tau) + 1]\right\} \tag{3.18}$$

$$-u_\ell(c_t, \ell_t) = u_c(c_t, \ell_t)w_t(1 - \tau) \tag{3.19}$$

$$r_t = f_K(k_t, \ell_t, z_t) \tag{3.20}$$

$$w_t = f_L(k_t, \ell_t, z_t) \tag{3.21}$$

$$\tau\left[w_t\ell_t + (r_t - \delta)k_t\right] = T_t \tag{3.22}$$

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.}(0, \sigma_z^2) \tag{3.23}$$

The state of the economy is defined by z_t and k_{t-1}. All other variables are jump variables. This gives us the following classifications:

$$
\begin{aligned}
X_t &= \{k_{t-1}\} \\
Y_t &= \{c_t, \ell_t, w_t, r_t, T_t\} \\
Z_t &= \{z_t\}
\end{aligned}
\tag{3.24}
$$

Our goal is the policy function:

$$X_{t+1} = \Phi(X_t, Z_t) \tag{3.25}$$

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3.7 The Steady State

Often, we need to find the steady state of the model before we can find the policy function. In a model with no stochastic shocks the steady state is easy to define. It is the state (or, more precisely, the value of the state variables) toward which the economy tends in the long run.

3.7.1 The Solow Model

A familiar example is the simple Solow growth model. In this model households save a constant portion of their income, savings and investment are equal and capital accumulates with depreciation.

$$S_t = sY_t$$
$$Y_t = K_t^\alpha L_t^{1-\alpha}$$
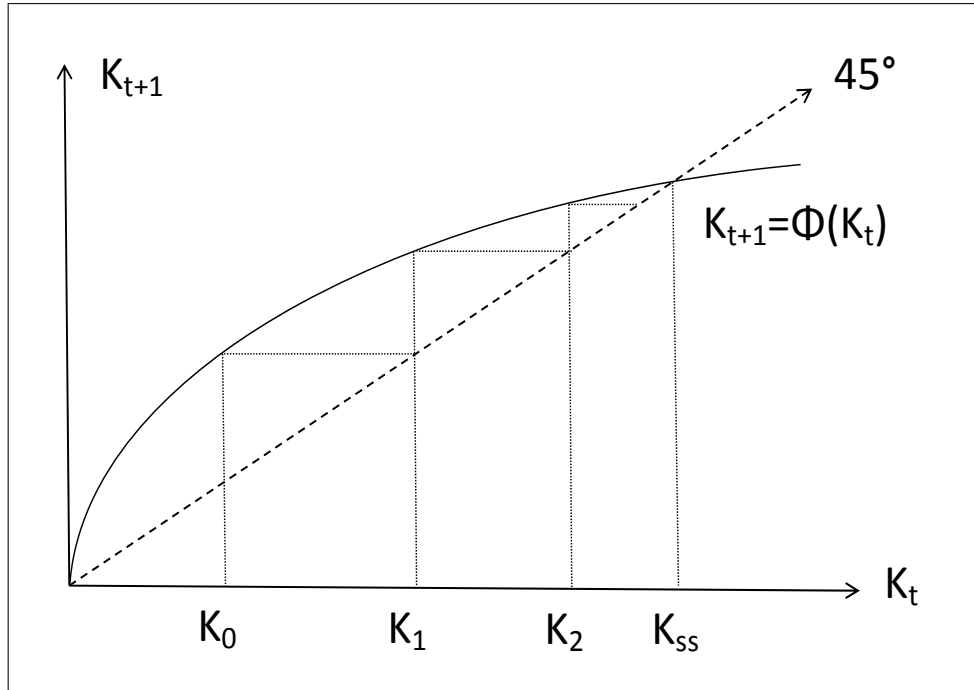$$L_t = \bar{L} = 1$$
$$S_t = I_t$$
$$I_t = K_{t+1} - (1-\delta)K_t$$

Iterative subsitution yields the following policy function:

$$K_{t+1} = (1-\delta)K_t + sK_t^\alpha \tag{3.26}$$

Figure 1 plots this policy function.

Suppose we started off the economy with a capital stock of K_0. The policy function, $K_{t+1} = \Phi(K_t)$ gives a value of K_1 for next period. Then,

Figure 1: Policy Function and Steady State for the Solow Model

next period, the policy function gives K_2, and so forth. In the long run the capital stock will converge to K_{ss}, the steady state value.

Since there are no stochastic shocks in this model, the state is described by a single variable, K_t. The dynamics of this model are such that the capital stock converges over time to its steady state value and then stays there forever.

If we allow for stochastic shocks, there is no steady state, at least not in the same sense. Suppose we have a different production function for the Solow model that allowed for a stochastic shock to productivity.

$$Y_t = K_t^\alpha (L_t e^{z_t})^{1-\alpha}$$

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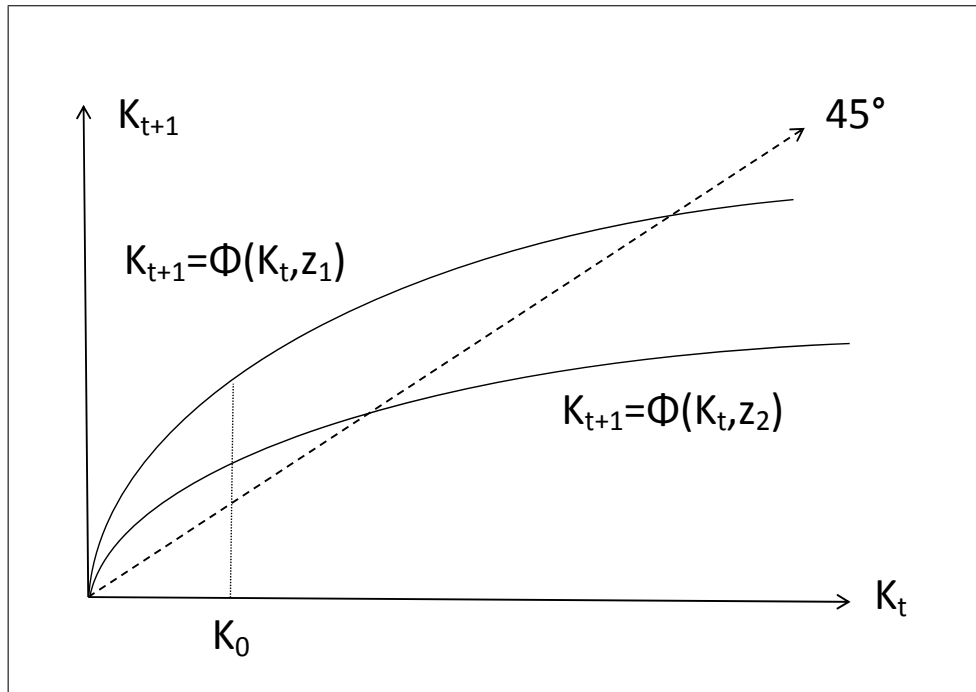Further suppose that the shock takes on one of two values randomly.

$$z_t = \begin{cases} z_1 & \text{with probability } \pi \\ z_2 & \text{with probability } 1 - \pi \end{cases}$$

The policy function becomes

$$K_{t+1} = (1 - \delta)K_t + sK_t^\alpha e^{(1-\alpha)z_t} \tag{3.27}$$

Figure 2 plots this. Because there are two policy functions and the econ-

Figure 2: Policy Function and Steady State for the Solow Model

omy switches randomly between these two, there is no steady state in the sense of a constant long-run value. However, in the long run, the state of

economy does settle down to a constant *distribution* for the capital stock. This distribution is referred to as the "ergodic" distribution and the economy is said to be in a "stationary" state.

Often we talk about the "steady state" of a stochastic model. When we do, we are referring to a counterfactual state that economy will not ever reach or maintain. This steady state is defined as the value for the state variables toward which the economy would trend, if the stochastic shock were permanently at its steady state value. Another way to think of this is that it is the steady state for a non-stochastic version of the model, where the innovations to the stochastic shocks (the ϵ^z's) are always zero.

To solve for the steady state we set all stochastic shocks to their long run value and do the same for the endogenous state variables.

In the stochastic Solow model we get:

$$\bar{K} = (1 - \delta)\bar{K} + s\bar{K}^\alpha e^{(1-\alpha)\bar{z}}; \quad \bar{z} = \pi z_1 + (1 - \pi)z_2 \qquad (3.28)$$

The solution for $\bar{K}$ is:

$$\bar{K} = \left(\frac{se^{(1-\alpha)\bar{z}}}{\delta} \right)^{\frac{1}{1-\alpha}} \qquad (3.29)$$

3.7.2 Steady State of the Baseline Model

To find the steady state in our DSGE model we need to proceed similarly and replace the time-period-specific values of $\{c_t, k_t, \ell_t, w_t, r_t, \tau_t \& z_t\}$ in (3.17) – (3.23) with their steady state values as below:

$$\bar{c} = (1 - \tau) \left[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k} \right] + \bar{T} \qquad (3.30)$$

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$$u_c(\bar{c}, \bar{\ell}) = \beta E_t \left\{ u_c(\bar{c}, \bar{\ell})[(\bar{r} - \delta)(1 - \tau) + 1] \right\} \qquad (3.31)$$

$$-u_\ell(\bar{c}, \bar{\ell}) = u_c(\bar{c}, \bar{\ell})\bar{w}(1 - \tau) \qquad (3.32)$$

$$\bar{r} = f_K(\bar{k}, \bar{\ell}, \bar{z}) \qquad (3.33)$$

$$\bar{w} = f_L(\bar{k}, \bar{\ell}, \bar{z}) \qquad (3.34)$$

$$\tau \left[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k} \right] = \bar{T} \qquad (3.35)$$

(3.30) – (3.35) implicitly define a steady state via a system of six equations in six unknowns, the unknowns being $\bar{k}, \bar{c}, \bar{w}, \bar{r}, \bar{\ell}$ and $\bar{T}$ (3.30) and (3.35) can be used as definitions, reducing the system to four equations in $\bar{k}, \bar{w}, \bar{r}$ and $\bar{\ell}$.

With this system we could perform comparative statics on the steady state by totally differentiating the remaining four equations: (3.31), (3.32), (3.33) and (3.34), using the chain rule on equations (3.30) and (3.35) as needed. Exercise 6 requires such a task.

To proceed further we need functional forms for $u(c, \ell)$ and $f(k, \ell, z)$. If we choose appropriate functional forms it will be possible to solve explicitly for the steady state value of k as a function of the parameters. Taking comparative statics is relatively easy for $\bar{k}$. Once this is known we can substitute it back into equations (3.30) – (3.35) appropriately to get the comparative statics for all the other steady state variables. This is the assignment for Exercise 9.

3.8 Additional Issues

Finally note that we have nowhere in our DSGE model defined other common variables such as investment and GDP. We can easily do so, however, by adding additional definitional equations. For example, GDP would be given by:

$$y_t = f(k_t, \ell_t, z_t) \tag{3.36}$$

Investment is given by:

$$i_t = k_{t+1} - (1 - \delta)k_t \tag{3.37}$$

In addition, while r_t is the rental rate on capital, it is not the same as the real interest rate that would be offered on financial assets. The closest analog is the *user cost of capital*, which is defined as:

$$u_t = r_t - \delta \tag{3.38}$$

Other variables of interest can be defined in similar ways. Once steady state values are found for the state variables (in our case $\bar{k}$ and $\bar{z}$) we can find the steady state values of all other variables in the model.

Also note that the state can be expanded to include variables that are not necessary to define the state. For example, if we cannot rewrite one of the equations so that ℓ_t is isolated as a definition, we would still have an equation or set of equations that implicitly define its value. In this case if often turns out to be useful to simply include ℓ_t in the set of endogenous state variables even though it is not strictly a state variable. Indeed we will

see that in some cases it is best to treat all the variables in the model as if they were state variables (this is the case with the DSGE modeling program, Dynare, for example). This means that it is perfectly legitimate to rewrite (3.24) as (3.39) or (3.40).

$$X_t = \{k_{t-1}, \ell_t\}$$
$$Y_t = \{c_t, w_t, r_t, T_t\} \tag{3.39}$$
$$Z_t = \{z_t\}$$

$$X_t = \{k_{t-1}, \ell_t, c_t, w_t, r_t, T_t\}$$
$$Y_t = \emptyset \tag{3.40}$$
$$Z_t = \{z_t\}$$

4 Brock and Mirman's Model

In this section we build a variant of the model in Brock and Mirman (1972). This model is absurdly simple, but is valuable for two reasons. First, the closed-form solution for the policy function is known. And second, it is an easy model to use when illustrating our numerical techiniques later in the course.

Households solve the followng dynamic program.

$$V(K_t, z_t) = \max_{K_{t+1}} \ln(e^{z_t} K_t^\alpha - K_{t+1}) + \beta E_t\{V(K_{t+1}, z_{t+1})\} \tag{4.1}$$

The associated Euler equation is:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}} \right\} \qquad (4.2)$$

The law of motion for z is:

$$z_{t+1} = \rho z_t + \sigma \varepsilon_t \qquad (4.3)$$

And the shocks, ε, are distributed as:

$$\varepsilon_t = \begin{cases} \sqrt{\frac{3}{2}} & \text{with probability } \frac{1}{3} \\ 0 & \text{with probability } \frac{1}{3} \\ -\sqrt{\frac{3}{2}} & \text{with probability } \frac{1}{3} \end{cases} \qquad (4.4)$$

You be we asked in a homework assignment to verify that the policy function takes the following form:

$$K_{t+1} = Ae^{z_t}K_t^\alpha \qquad (4.5)$$

You will need to find the value of A as a function of the model's parameters.

The steady state value of K is defined by using (4.5) with $K_t = K_{t+1}$ and $z_t = 0$.

$$\bar{K} = A^{\frac{1}{1-\alpha}} \qquad (4.6)$$

The parameters of the model are, α, β, ρ and σ. When values are needed for numerical calculations use the following default values unless otherwise

indicated:

$$\alpha = .35$$
$$\beta = .98$$
$$\rho = .95$$
$$\sigma = .02$$

5   Introduction to Solution Methods

Except for a few known special cases, it is generally impossible to solve algebraically for the policy function in a DSGE model. One case where it is possible is given in Homework 1. This means we must resort to numerical techniques to solve and simulate DSGE models. The set of solution techniques that have been used in the past includes:

- discrete grid approximations

- spline function approximations

- linear-quadratic approximations objective functions and contstraints

- log-linearzation of the characterizing equations

- higher-order approximation of the characterizing equations

We will discuss all of these, but will go into special detail with the last two methods

5.1 Grids and Splines

Discrete grid methods are very computationally intensive, but can be ideal for highly nonlinear policy functions. In our DSGE model the policy function is $k_{t+1} = \Phi(k_t, z_t)$. To approximate this with a discrete grid, we would set up a support for k and z with a finite number of elements (the grid). To find the policy function, for every combination of the values for k_t and z_t in the grid, we would find the value of k_{t+1} in the finite support of k that comes cloeset to satisfying the general equilibrium conditions. If there are N elements in the support of k and M elements in the support of z, then the policy function is an $N \times M$ array with an index number in each element of the array indicating which of the N elements in the support of k is optimal. Figure 3 illustrates this discretized policy function for a particular value of z, z_0.

As the grid becomes finer with more elements in the supports, the approximation becomes better.

A spline approximation allows us to fill in the values between the discrete points on the support. Simple splines could be linear, or they could be curved to approximate smooth functions better using a quadratic or cubic function over the interval between the points.

There is a great deal of literature on the optimal grid structure and the optimal splines for approximation in a variety of settings. Engineering applications often use these tools. In economics they are less widely used, most likely because the behavior of many DSGE models is quite smooth making other approximation techniques computationally more efficient.

Figure 3: Policy Function and Steady State for the Solow Model
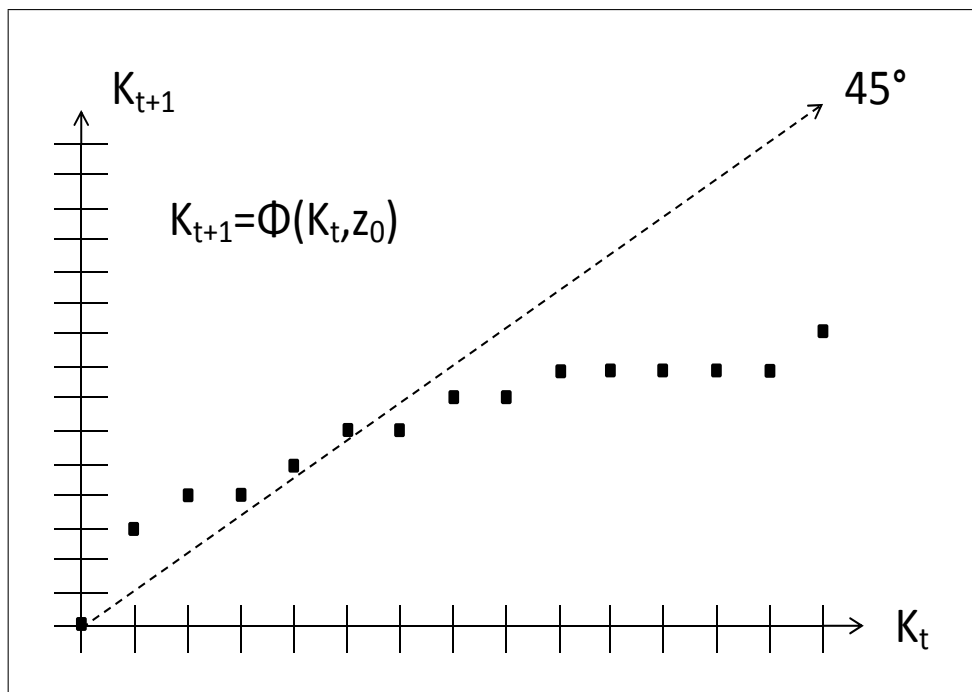
5.2 Blanchard-Kahn and Linear-Quadratic Methods

One of the first approximation methods to be used with DSGE models was introduced by Blanchard and Kahn (1980) who adapted numerical techniques from the engineering literature. An excellent discussion of this is given in McCandless (2008, pp. 129-132).

Another early method employed by Kydland and Prescott (1982) is to optimize an objective function that is approximated by a quadratic, subject to linear constraints. This is usually the household's utility function maximized subject to a budget constraint. Intutively, since the objective is quadratic, the Euler equations derived will be linear yielding highly tractable equations

from which to derive a solution. In terms of our baseline DSGE model we take a quadratic approximation of the utility function in (3.1) and then use the first-order condtions to derive specific versions of equations (3.7) and (3.8). These will be simple linear equations and the system can be solved following the steps outlined in McCandless (2008, ch. 7).

5.3   Log-Linearization Methods

The best sources here are Uhlig (1999) and Christiano (2002). Log-linearizartion will be covered in excruciating detail in section 6. Here we will simply note the intuition. Rather than approximate the objective function as above, log-linearization techniques involve taking approximations of the final characterizing equations. In terms of our DSGE model from 3 we would take logs of equations (3.17) – (3.11) and then take linear approximations of these, almost always by taking a first-order Taylor-series expansion of the equations. If we assume the policy function is also log-linear then some straightforward algebra yields its coefficients. Numerically solving for these coefficients can be challenging, particularly if the state space of the model is large.

The Taylor-series expansion requires a point about which the functions are approximated. In almost all cases this is the steady state of the model. Most models are stable and well-behaved and the stationary state is one where the state fluctuates around in a neighborhood near the zero-innovation steady state. Evans and Phillips (2012) show how linearizing about a different point may improve the approximation for models that are unstable or which are for other reasons rarely in the neighborhood of the steady state.

5.4 Perturbation Methods

Log-linearization methods will be very good approximations in the neighborhood of the expansion point. One way to improve the accuracy of the approximation is to incorporate higher-order terms from the Taylor-series expansion of the characterizing equations. Higher-order terms means finding greater numbers of coefficients. The classic citation for the numerical method of finding higher-order coefficients is Judd (1992) and the method is referred to a the "pertubation method."

Consider our intertemporal Euler equation from (3.7), reproduced below.

$$u_c(c_t, \ell_t) = \beta E_t \left\{ u_c(c_{t+1}, \ell_{t+1})[(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \qquad (5.1)$$

Note that the right-hand side of the equation involves taking an expectation. If we linearize or log-linearize this equation, the expectations operater passes through and we end up with linear equations in the expected values of the variables. This generates a phenomenon known as "certainty equivalence" which will not hold generally. The right-hand side takes the expectation of the marginal utility of consumption times a rate of return on capital. By log-linearizing this becomes equal to the marginal utility of the expected consumption times the expected return on capital. These are only approximately the same. And the difference can be important if we are dealing with things like risk aversion and risk premia. For this reason, higher-order approximations are a must when dealing with models where riskiness is a central feature of the model.

5.5 Sparse Grid and GSSA

More recently Krueger and Kubler (2004) and Judd, Maliar, and Maliar (2011) have used computational techniques that give very good approximations to policy functions. These techniques do not require the imposition of certainty equivalence and are suitable for a large variety of dynamic general equilibrium models.

6 Log-Linear Approximations

Log-linear approximations are a very common way of solving and simulating DSGE models. Recently, due to the easy availability of the software package, Dynare, higher-order approximations have begun to replace these methods. The methods are nonetheless useful as a step to understanding higher-order approximations. In many cases the computational simplicity of these approximations may be worth the loss of accuracy relative to the higher-order methods.

6.1 A General Method

Let us first present log-linearization in a very general way. We follow the notation in Uhlig (1999) in the discussion that follows. Once we understand this in general, we will return to our baseline DSGE model and use it as an example.

Let us group our model's variables into two groups as above. Z_t is a vector of exogenous state variables. X_t will be a vector of endogenous state variables. Recall we can include non-state variables in this vector without

loss of generality as long as there exists a subset of the elements in X_t and Z_t that is sufficient to describe the state of the economy. Let n_X be the cardinality of X_t and n_Z be the cardinality of Z_t.

We will assume that Z_t follows first-order vector autoregression for its law of motion.

$$Z_t = (I - N)\bar{Z} + NZ_{t-1} + \varepsilon_t \tag{6.1}$$

N is a $n_Z \times n_Z$ square matrix.

We will take the characterizing equations for the model and write them as a vector of functions in the following form:

$$E_t\{\Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t)\} = 1 \tag{6.2}$$

X_{t+1}, X_t and X_{t-1} are $n_X \times 1$ vectors. Z_{t+1} and Z_t are $n_Z \times 1$ vectors and Γ outputs a $n_X \times 1$ vector.

We next take the natural logarithm of all these functions and apply a Taylor-series expansion to them, expanding about the steady state values of X and Z. We keep only the constants and linear terms and discard all higher-order terms in the expansion. Because we have discarded these terms the equations we have derived are only approximations.

$$E_t\left\{\ln\Gamma(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z}) + \sum_{i\in\mathcal{I}} \frac{\Gamma_i(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}{\Gamma(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}(i - \bar{i})\right\} = 0 \tag{6.3}$$

where $\mathcal{I} = \{X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t\}$ and Γ_i denotes the matrix of derivatives of the vector of functions Γ with respect to element i in $\mathcal{I}$.

Since the steady state values of X and Z satisfy (6.2), the first term in (6.3) is zero. We can also divide each term in the sum by the vector of variables for which the derivative is taken to get:

$$E_t \left\{ \sum_{i \in \mathcal{I}} \frac{\bar{i} \Gamma_i(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}{\Gamma(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})} \frac{i - \bar{i}}{\bar{i}} \right\} = 0 \qquad (6.4)$$

We can write this in a more compact form using Uhlig's notation as:

$$E_t \left\{ F \tilde{X}_{t+1} + G \tilde{X}_t + H \tilde{X}_{t-1} + L \tilde{Z}_{t+1} + M \tilde{Z}_t \right\} = 0 \qquad (6.5)$$

where a tilde over a variable indicates its percent deviation from its steady state value for X, i.e. $\tilde{X}_t \equiv \frac{X_t - \bar{X}}{\bar{X}}$, and an absolute deviation from the steady state value for Z, i.e. $\tilde{Z}_t \equiv Z_t - \bar{Z}$. The coefficients are matrices of derivatives evaluated at the steady state values. For example, $F \equiv \frac{\bar{X} \Gamma_{X_{t+1}}(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}{\Gamma(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}$, and $L \equiv \frac{\Gamma_{Z_{t+1}}(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}{\Gamma(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})}$. F, G and H are $n_X \times n_X$ square matrices, while L and M are $n_X \times n_Z$.

With this notation in place, we can also rewrite (6.1) as:

$$\tilde{Z}_t = N \tilde{Z}_{t-1} + \varepsilon_t \qquad (6.6)$$

We now hypothesize that the transition function, $X_t = \Phi(X_{t-1}, Z_t)$ can also be log-linearly approximated by

$$\tilde{X}_t = P \tilde{X}_{t-1} + Q \tilde{Z}_t \qquad (6.7)$$

where P is a $n_X \times n_X$ square matrix and Q is $n_X \times n_Z$.

We know the values of F, G, H, L and M since these are functions of the model parameters and steady state values (which are also functions of the parameters). N can be considered as part of the parameter set. We do not know the values of P and Q, however.

We can solve for them by iteratively substituting appropriate versions of (6.7) and (6.6) into (6.5).

After some tedious matrix algebra (see Homework 10) this reduces to:

$$[(FP + G)P + H]\tilde{X}_{t-1} + [(FQ + L)N + (FP + G)Q + M]\tilde{Z}_t = 0 \quad (6.8)$$

(6.8) is true for all values of X_{t-1} and Z_t. This means that the coefficients on these terms in the equation must be zero. In turn, this gives us two equations that implicitly define P and Q.

$$FP^2 + GP + H = 0 \quad (6.9)$$

$$FQN + (FP + G)Q + (LN + M) = 0 \quad (6.10)$$

Condition (6.9) requires solving a matrix quadratic in P. Once P is known, condition (6.10) can be used to solve for a value of Q.

6.2 Analytical Log-Linearization

Before proceeding to our particular model, let's first discuss how one log-linearizes characteristic equations analytically.

Suppose we have a characterizing equation like that below.

$$E_t \left\{ \frac{x_t^b}{x_{t-1}^d} \right\} = 1$$

We can substitute in the following identities: $x_t = \bar{x}(1 + \tilde{x}_t) \cong \bar{x}e^{\tilde{x}_t}$ and $x_{t-1} = \bar{x}e^{\tilde{x}_{t-1}}$ to get:

$$E_t \left\{ \frac{\bar{x}e^{b\tilde{x}_t}}{\bar{x}e^{d\tilde{x}_{t-1}}} \right\} = E_t \left\{ e^{b\tilde{x}_t - d\tilde{x}_{t-1}} \right\} = 1$$

Noting that while x_{t-1} is known and we need not take expectations, but that this is not so for x_t and taking natural logs:

$$0 = \ln E_t \left\{ e^{b\tilde{x}_t - d\tilde{x}_{t-1}} \right\}$$
$$\cong \ln E_t \left\{ 1 + b\tilde{x}_t - d\tilde{x}_{t-1} \right\}$$
$$= \ln(1 + bE_t\{\tilde{x}_t\} - d\tilde{x}_{t-1})$$
$$\cong bE_t\{\tilde{x}_t\} - d\tilde{x}_{t-1}$$

Suppose that a different characterizing equation is

$$E_t \left\{ x_t^b + f x_{t-1}^d \right\} = 1$$

Proceeding as before:

$$
\begin{aligned}
0 &= \ln E_t \left\{ \bar{x} e^{b\tilde{x}_t} + f \bar{x} e^{d\tilde{x}_{t-1}} \right\} \\
&= \ln E_t \left\{ \bar{x}(e^{b\tilde{x}_t} + f e^{d\tilde{x}_{t-1}}) \right\} \\
&\cong \ln E_t \left\{ \bar{x}[1 + b\tilde{x}_t + f(1 + d\tilde{x}_{t-1})] \right\} \\
&= \ln \bar{x} + \ln[1 + f + b E_t\{\tilde{x}_t\} + f d\tilde{x}_{t-1})]
\end{aligned}
$$

We can proceed similarly with almost any characterizing equation.

6.3 Numerical Log-Linearization

Another way to proceed is to use numerical methods to log-linearize. Since the coefficients of (6.5) are functions of derivatives evaluated at the steady state. We can find these numerically as well as analytically.

The procedure for numerical deriviatives is as follows:

- Evalate Γ at the steady state values. Denote this column vector $\bar{\Gamma}$.

- One-by-one disturb each element in $[X_t \ Z_t]$ by a small amount, ε, and evaluate Γ again at each of these values. This will give a $n_X \times 1$ column vector for each of the $3n_X + 2n_Z$ values. Denote each of these vectors Γ_j, where j indexes the elements of $\Theta \equiv [X_{t+1} \ X_t \ X_{t-1} \ Z_{t+1} \ Z_t]$.

- Find the derivative of for the log of Γ_{ij} (i indexing the row) by using $D_{ij} = \frac{\Theta_j(\Gamma_{ij} - \bar{\Gamma}_i)}{\varepsilon \bar{\Gamma}_i}$ for the Xs and $D_{ij} = \frac{\Gamma_{ij} - \bar{\Gamma}_i}{\varepsilon \bar{\Gamma}_i}$ for the Zs. This generates a large matrix of derivatives, D, that is $n_X \times (3n_X + 2n_Z)$. Our coefficients come from subsets of this matrix. For example, F is all rows and columns 1 through n_X of D.

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6.4 Linearizing Brock and Mirman's Model

Let us first solve for the linearized approximation, not the log-linearized one. Recall the Euler equation from the Brock and Mirman model.

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}} \right\}$$

We rewrite this equation and put it in the form of (6.2).

$$E_t \left\{ \beta[e^{z_{t+1}}K_{t+1}^{\alpha-1}(e^{z_t}K_t^\alpha - K_{t+1}) - e^{z_{t+1}}K_{t+1}^\alpha + K_{t+2}] \right\} = 0 \qquad (6.11)$$

By differentiating (6.2) with respect to K_{t+2}, K_{t+1}, K_t, z_{t+1} and z_t we can recover the Uhlig matrices:

$$
\begin{aligned}
F =& \beta \\
G =& \beta e^{z_{t+1}}K_{t+1}^{\alpha-1} \left[K_{t+1}^{-1}(\alpha-1)(e^{z_t}K_t^\alpha - K_{t+1}) - 1 - \alpha \right] \\
H =& \beta e^{z_{t+1}}K_{t+1}^{\alpha-1}e^{z_t}K_t^{\alpha-1}\alpha \\
L =& \beta \left[e^{z_{t+1}}K_{t+1}^{\alpha-1}(e^{z_t}K_t^\alpha - K_{t+1}) - (e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}) \right] \\
M =& \beta e^{z_{t+1}}K_{t+1}^{\alpha-1}e^{z_t}K_t^{\alpha-1}
\end{aligned}
$$

Evaluating these at their steady state values gives:

$$F = \beta$$
$$G = \beta \bar{K}^{\alpha-1} \left[\bar{K}^{-1}(\alpha-1)(\bar{K}^\alpha - \bar{K}) - 1 - \alpha \right]$$
$$H = \beta \bar{K}^{2(\alpha-1)} \alpha$$
$$L = \beta \left[\bar{K}^{\alpha-1}(\bar{K}^\alpha - \bar{K}) - (\bar{K}^\alpha - \bar{K}) \right]$$
$$M = \beta \bar{K}^{2(\alpha-1)}$$

To evaluate (6.15) recall $\bar{K} = A^{\frac{1}{1-\alpha}}$. We can then use equations (6.9) and (6.10) to derive the scalar values P and Q.

$$P = \frac{-G \pm \sqrt{G^2 - 4FH}}{2F} \tag{6.12}$$

$$Q = -\frac{LN + M}{FN + FP + G} \tag{6.13}$$

In this case, since we have linearized (not log-linearized) the policy function is:

$$K_{t+1} = \bar{K} + P(K_t - \bar{K}) + Q z_t \tag{6.14}$$

6.5   Log-Linearizing Brock and Mirman's Model

To log-linearize we must replace K_t with $\bar{K} e^{\tilde{K}_t}$. We do this for (6.11) and obtain:

$$E_t \left\{ \beta [\bar{K}^{\alpha-1} e^{z_{t+1} + (\alpha-1)\tilde{K}_{t+1}} (\bar{K}^\alpha e^{z_t + \alpha \tilde{K}_t} - \bar{K} e^{\tilde{K}_{t+1}}) - \bar{K}^\alpha e^{z_{t+1} + \alpha \tilde{K}_{t+1}} + \bar{K} e^{\tilde{K}_{t+2}}] \right\} = 0$$

We rewrtite this as:

$$E_t \left\{ \begin{array}{c} \bar{K}^{-1} e^{z_{t+1} + (\alpha-1)\tilde{K}_{t+1} + z_t + \alpha \tilde{K}_t} \\ -\bar{K}^\alpha e^{z_{t+1} + (\alpha-1)\tilde{K}_{t+1} + \tilde{K}_{t+1}} \\ -\bar{K}^\alpha e^{z_{t+1} + \alpha \tilde{K}_{t+1}} \\ +\bar{K} e^{\tilde{K}_{t+2}} \end{array} \right\} = 0$$

We rewrtite this further as:

$$E_t \left\{ \begin{array}{c} \bar{K}^{-1}(1 + z_{t+1} + (\alpha - 1)\tilde{K}_{t+1} + z_t + \alpha \tilde{K}_t) \\ -\bar{K}^\alpha(1 + z_{t+1} + (\alpha - 1)\tilde{K}_{t+1} + \tilde{K}_{t+1}) \\ -\bar{K}^\alpha(1 + z_{t+1} + \alpha \tilde{K}_{t+1}) \\ +\bar{K}(1 + \tilde{K}_{t+2}) \end{array} \right\} = 0$$

Note that the steady state version of (6.11) implies that $\bar{K}^{-1} - \bar{K}^\alpha - \bar{K}^\alpha + \bar{K} = 0$. This allows us to again rewrite as:

$$E_t \left\{ \begin{array}{c} \bar{K}^{-1}(z_{t+1} + (\alpha - 1)\tilde{K}_{t+1} + z_t + \alpha \tilde{K}_t) \\ -\bar{K}^\alpha(z_{t+1} + (\alpha - 1)\tilde{K}_{t+1} + \tilde{K}_{t+1}) \\ -\bar{K}^\alpha(z_{t+1} + \alpha \tilde{K}_{t+1}) \\ +\bar{K}(\tilde{K}_{t+2}) \end{array} \right\} = 0$$

Regrouping terms:

$$
E_t \left\{
\begin{array}{c}
(\bar{K})\tilde{K}_{t+2} \\
(\bar{K}^{-1}(\alpha - 1) - 2\bar{K}^{\alpha}\alpha)\tilde{K}_{t+1} \\
(\bar{K}^{-1}\alpha)\tilde{K}_t \\
(\bar{K}^{-1} - 2\bar{K}^{\alpha})z_{t+1} \\
(\bar{K}^{-1})z_t
\end{array}
\right\} = 0
$$

This gives:

$$
\begin{aligned}
F &= \bar{K} \\
G &= \bar{K}^{-1}(\alpha - 1) - 2\bar{K}^{\alpha}\alpha \\
H &= \bar{K}^{-1}\alpha \\
L &= \bar{K}^{-1} - 2\bar{K}^{\alpha} \\
M &= \bar{K}^{-1}
\end{aligned}
$$

6.6   Log-Linearizing Our Baseline Model

Recall the model defined by (3.17) – (3.23). $Z_t = \{z_t\}$, but we need to determine what variables are in the vector X_t. We certainly want to include k_{t-1}, but what jump variables should we include?

As we mentioned in section 3.8, we could include all the jump variables. However, let's try to be a bit more conservative with the list. Any variables that can be isolated as definitions will not be included as elements of X_t. (3.17), (3.20), (3.21), and (3.11) give us four such definitions.

$$r_t = f_K(k_t, \ell_t, z_t)$$

$$w_t = f_L(k_t, \ell_t, z_t)$$

$$T_t = \tau \left[w_t \ell_t + (r_t - \delta)k_t \right]$$

$$c_t = (1 - \tau) \left[w_t \ell_t + (1 + r_t - \delta)k_t \right] + T_t - k_{t+1}$$

(6.15)

So let's define $X_t = \{k_{t-1}, \ell_t\}$, even though ℓ_t is not strictly a state variable.

If we use $\ln c_t + a \ln(1 - \ell_t)$ as the utility function. Then, $u_c(c_t, \ell_t) = c_t^{-1}$ and $u_\ell(c_t, \ell_t) = -(1 - \ell)_t^{-1}$. Our characterizing equations from (3.18) and (3.19) become:

$$E_t \left\{ \beta \frac{c_t}{c_{t+1}} (1 + r_{t+1} - \delta)(1 - \tau) \right\} = 1 \qquad (6.16)$$

$$\frac{c_t}{w_t(1 - \tau)(1 - \ell_t)} = 1 \qquad (6.17)$$

The law of motion for Z is the same as (3.23).

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \varepsilon_t^z; \quad \varepsilon_t^z \sim \text{i.i.d.}(0, \sigma_z^2) \qquad (6.18)$$

To implement the procedure from section 6.1 above we need transform (6.18) and write it in terms of the deviations of z_t from its steady state value:

$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + \varepsilon_t^z \qquad (6.19)$$

We next need to take the natural logs of (6.16) and (6.17) and then

linearize them. We can do this analytically or numerically. Let's do it analytically here.

First let's choose a functional for for the production function. We will use $Y_t \equiv e^{z_t} k_t^\alpha \ell_t^{1-\alpha}$.

Let's log-linearize our definitions in (6.15) first (see the Appendix for details).

$$\tilde{r}_t = \tilde{z}_t + (\alpha - 1)\tilde{k}_t + (1 - \alpha)\tilde{\ell}_t \tag{6.20}$$

Proceeding similarly with the other lines gives:

$$\tilde{w}_t = \tilde{z}_t + \alpha\tilde{k}_t - \alpha\tilde{\ell}_t \tag{6.21}$$

$$\tilde{T}_t = \frac{\tau\bar{w}\bar{\ell}}{\bar{T}}\tilde{w}_t + \frac{\tau\bar{w}\bar{\ell}}{\bar{T}}\tilde{\ell}_t + \frac{(1+\bar{r}-\delta)\bar{k}}{\bar{T}}\tilde{r}_t + \frac{(1+\bar{r}-\delta)\bar{k}}{\bar{T}}\tilde{k}_t \tag{6.22}$$

$$\begin{aligned} c_t = & \frac{(1-\tau)\bar{w}\bar{\ell}}{\bar{c}}\tilde{w}_t + \frac{(1-\tau)\bar{w}\bar{\ell}}{\bar{c}}\tilde{\ell}_t + \frac{(1+\bar{r}-\delta)\bar{k}}{\bar{c}}\tilde{r}_t... \\ & + \frac{(1+\bar{r}-\delta)\bar{k}}{\bar{c}}\tilde{k}_t + \frac{\bar{T}}{\bar{c}}\tilde{T}_t - \frac{\bar{k}}{\bar{c}}\tilde{k}_{t+1} \end{aligned} \tag{6.23}$$

Log-linearizing the characterizing equations in (6.16) and (6.17) gives:

$$\tilde{c}_t - \tilde{c}_{t+1} + \tilde{r}_{t+1} = 0 \tag{6.24}$$

$$\tilde{c}_t - \tilde{w}_t + \tilde{\ell}_t = 0 \tag{6.25}$$

Substituting (6.20) – (6.23) into (6.24) and (6.25) gives the elements of the coefficients in (6.5) as functions of parameters and steady state values.

If we had chosen $X_t = \{k_{t-1}, \ell_t, r_t, w_t, T_t, c_t\}$ then (6.20) – (6.25) would give us the coefficients.

6.7   Simulating Log-Linearized DSGE Modes

We begin our simulation at a set of starting values, $[X_0 \; Z_1]$. We need to first convert these to deviations from the steady state. For values of X we take the percent deviation, $\tilde{X}_0 = \ln X_0 - \ln \bar{X}$. For values of Z we take the absolute deviation, $\tilde{Z}_0 = Z_0 - \bar{Z}$

Recall the law of motion for Z_t is (6.6).

$$\tilde{Z}_t = N \tilde{Z}_{t-1} + \varepsilon_t$$

We can use this to generate a history of $\tilde{Z}$s independent of the rest of the economy, since the $\tilde{Z}$s are exogenous to the model. We do so by generating a history of ε's using a random number generator. We start with $t = 2$ and iteratively apply (6.6) to generate the $\tilde{Z}$ time series.

Recall our log-linearized policy function is equation (6.7).

$$\tilde{X}_t = P \tilde{X}_{t-1} + Q \tilde{Z}_t$$

We can use this to iteratively generate a history of $\tilde{X}$s starting with $t = 1$.

Once we have full histories for $[\tilde{X}_t \; \tilde{Z}_t]$ we can convert these back to actual levels by using $X_t = \bar{X} e^{\tilde{X}_t}$ and $Z_t = \tilde{Z}_t + \bar{Z}$.

A   Appendix

The details of the algebra required to obtain (6.20) follow:

$$r_t = e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}$$

$$\bar{r}e^{\tilde{r}_t} = e^{\bar{z}+\tilde{z}_t}(\bar{k}e^{\tilde{k}_t})^{(\alpha-1)}\overline{(\ell e^{\tilde{\ell}_t})}^{(1-\alpha)}$$

$$\bar{r}(1+\tilde{r}_t) = e^{\bar{z}}(1+\tilde{z}_t)\bar{k}^{\alpha-1}[1+\tilde{k}_t(\alpha-1)]\bar{\ell}^{1-\alpha}[1+\tilde{\ell}_t(1-\alpha)]$$

$$\bar{r}+\bar{r}\tilde{r}_t = e^{\bar{z}}\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha}(1+\tilde{z}_t)[1+\tilde{k}_t(\alpha-1)][1+\tilde{\ell}_t(1-\alpha)]$$

$$\bar{r}+\bar{r}\tilde{r}_t = e^{\bar{z}}\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha}[1+\tilde{z}_t+\tilde{k}_t(\alpha-1)+\tilde{\ell}_t(1-\alpha)+\Xi]; \quad \Xi \cong 0$$

$$\bar{r}+\bar{r}\tilde{r}_t = e^{\bar{z}}\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha} + e^{\bar{z}}\bar{k}\bar{\ell}[\tilde{z}_t+\tilde{k}_t(\alpha-1)+\tilde{\ell}_t(1-\alpha)]$$

$$\text{noting } \bar{r} = e^{\bar{z}}\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha}$$

$$\bar{r}\tilde{r}_t = \bar{r}[\tilde{z}_t+\tilde{k}_t(\alpha-1)+\tilde{\ell}_t(1-\alpha)]$$

$$\tilde{r}_t = \tilde{z}_t + (\alpha-1)\tilde{k}_t + (1-\alpha)\tilde{\ell}_t \tag{A.1}$$

Exercises

Homework 1

For the Brock and Mirman model, find the value of A in the policy function. Show that your value is correct.

For this case find an algebraic solution for the policy function, $k_{t+1} = \Phi(k_t, z_t)$. Couple of good sources for hints are Stokey, Prescott, and Lucas (1989, excercise 2.2, p. 12) and Sargent (1987, exercise 1.1, p. 47).

Homework 2

For the model in section 3 of the notes consider the following functional forms:

$$u(c_t, \ell_t) = \ln c_t + a \ln (1 - \ell_t)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms.

Can you use the same tricks as in homework 1 to solve for the policy function in this case? Why or why not?

Homework 3

For the model in section 3 of the notes consider the following functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \ln (1 - \ell_t)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms.

Homework 4

For the model in section 3 of the notes consider the following functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi}$$

$$F(K_t, L_t, z_t) = e^{z_t} \left[\alpha K_t^{\eta} + (1 - \alpha) L_t^{\eta} \right]^{\frac{1}{\eta}}$$

Write out all the characterizing equations for the model using these functional forms.

Homework & Lab 5a

For the model in section 3 of the notes consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$$

$$F(K_t, L_t, z_t) = K_t^\alpha (L_t e^{z_t})^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms. Assume $\ell_t = 1$.

Write out the steady state versions of these equations. Solve algebraically for the steady state value of k as a function of the steady state value of z and the parameters of the model. Numerically solve for the steady state values of all variables using the following parameter values: $\gamma = 2.5$, $\beta = .98$, $\alpha = .40$, $\delta = .10$, $\bar{z} = 0$ and $\tau = .05$. Also solve for the steady state values of output and investment. Compare these values with the ones implied by the algebraic solution.

Homework & Lab 5b

For the model in section 3 of the notes consider the following functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a\frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi}$$

$$F(K_t, L_t, z_t) = K_t^\alpha (L_t e^{z_t})^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms. Write out the steady state versions of these equations. Numerically solve for the steady state values of all variables using the following parameter values: $\gamma = 2.5$, $\xi = 1.5$, $\beta = .98$, $\alpha = .40$, $a = .5$, $\delta = .10$, $\bar{z} = 0$, and $\tau = .05$. Also solve for the steady state values of output and investment.

Homework 6

For the steady state in section 3.7 of the notes use total differentiation to solve for the full set of comparative statics and sign them where possible. Find $\frac{\partial y}{\partial x}$ for $y \in \{\bar{k}, \bar{w}, \bar{r}, \bar{\ell}\}$ and $x \in \{\delta, \tau, \bar{z}\}$.

Assume $f_K > 0$, $f_{KK} < 0$, $f_L > 0$, $f_{LL} < 0$, $u_c > 0$, $u_{cc} < 0$, $u_\ell < 0$ and $u_{\ell\ell} > 0$.

Note this problem gets very tedious. Solving via MAPLE would be a wonderful shortcut and you are welcome to proceed that way. However, you may also just shown how you could set the problem up using linear algebra and show what steps need to be taken to get the comparative statics, but not actually perform the algebra.

Lab 7a

For the Brock and Mirman model in section (4) set up a discrete grid for K with 100 values ranging from .0001 to $5\bar{K}$. Also set up a discrete grid for z with 100 values ranging from -5σ to $+5\sigma$. Set up a value function array, V that stores the value for all 10,000 possible permuations of K and z. Also set up a policy function array, H, that stores the optimal index value of K' for all all 10,000 possible permuations of K and z.

To begin assume that all elements of V are zero and that all elements of H point to the lowest possible value for K (.0001).

Loop over all possible values of K and z and for each combination find 1) the optimal value of K' from the 100 possible values. Store this value in an updated policy function array, H_{new}. Also find 2) the value implied by this choice given the current value function. Store this in an updated value function array, V_{new}.

Once this is completed for all K and z check to see if V is approzimately equal to V_{new}. If so, output the value function and policy function arrays. If not, replace V with V_{new} and H with H_{new} and repeat the search above.

When finished plot the three-dimensional surface plot for the policy function $K' = H(K, z)$. Compare this with the closed form solution from the notes.

Lab 7b

Repeat the above exercise using $k \equiv \ln K$ in place of K as the endogenous state variable.

Lab 8a

For the Brock and Mirman model in section (4) use Uhlig's notation to analytically find the values of the following matrices: F, G, H, L, M & N as functions of the parameters. Given these find the values of P&Q, also as functions of the parameters. Imposing our calbrated parameter values, plot the three-dimensional surface plot for the policy function $K' = H(K, z)$.

Compare this with the closed form solution fromthe notes and the solution you found using the grid search method.

Lab 8b

Repeat the above exercise using $k \equiv \ln K$ in place of K as the endogenous state variable.

Lab 9

For the baseline modle in section (3) numerically, find $\frac{\partial y}{\partial x}$ for $y \in \{\bar{k}, \bar{w}, \bar{r}, \bar{\ell}\}$ and $x \in \{\delta, \tau, \bar{z}\}$ assuming $u(c_t, \ell_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi}-1}{1-\xi}$ and $F(K_t, L_t, z_t) = K_t^\alpha (L_t e^{z_t})^{1-\alpha}$. Use the following parameter values: $\gamma = 2.5$, $\xi = 1.5$, $\beta = .98$, $\alpha = .40$, $a = .5$, $\delta = .10$, $\bar{z} = 0$, and $\tau = .05$.

Homework 10

Do the necessary tedious matrix algebra necessary to transform equation (6.5) into equation (6.8).

Lab 11

Assume $n_X = n_Z = 1$. Show how to solve for the elements of P and Q. Write a program in Python that will do this given F, G, H, L, M and N as inputs.

Now assume n_X and n_Z take on arbitrary values greater than one. Show how to solve for the elements of P and Q. Write a program in Python that will do this given F, G, H, L, M and N as inputs.

The MATLAB programs for Uhlig (1999) are available online and may be of some help structuring the Python code. The discussion in the paper itself may also be a big help.

Homework & Lab 12

Suppose that instead of including the jump variables of our our model in the vector X_t above, we separated them into a separate vector, Y_t. In this case we can linearize our system into the following equations:

$$0 = A\tilde{X}_t + B\tilde{X}_{t-1} + C\tilde{Y}_t + D\tilde{Z}_t$$
$$0 = E_t\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + J\tilde{Y}_{t+1} + K\tilde{Y}_t + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\}$$
$$\tilde{Z}_t = N\tilde{Z}_{t-1} + \varepsilon_t$$

Assume that C is of full rank. By once again hypothesizing that the transition functions for the model are log-linear, that is they are of the form:

$$\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$$
$$\tilde{Y}_t = R\tilde{X}_{t-1} + S\tilde{Z}_t$$

Derive algebraic solutions for P, Q, R, and S using techniques similar to those in the notes. Modify the program you wrote in Homework 9 slightly to accommodate separating jump variables from state variables. What advantages might this approach have computationally? In setting up the matricies describing the model?

Lab 13a

Assume $n_X = n_Z = 1$. Write a Python program that will find the numerical values for the derivative of the characterizing equation, taking the parameters as given. Note that you will need to find the steady state value of X first.

Now assume n_X and n_Z take on arbitrary integer values greater than one. Write a Python program that will find the numerical values for the derivatives of the characterizing equations in this case.

Lab 13b

Repeat the second part of the excercise above, where n_X and n_Z take on arbitrary integer values greater than one. However now take numerical values for the the derivatives of the natural logarthims of the characterizing equations.

Lab 14

For the log-linearized model in section 6.6 of the notes consider the following fundamental functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a\frac{(1 - \ell_t)^{1-\gamma} - 1}{1 - \gamma}$$
$$F(K_t, L_t, z_t) = K_t^\alpha (L_t e^{z_t})^{1-\alpha}$$

Use the following paramter values: $\gamma = 2.5$, $\beta = .98$, $\alpha = .40$, $a = .5$, $\delta = .10$, $\bar{z} = 0$, $\rho_z = .9$ and $\tau = .05$.

Find the values of P and Q in this case if $X_t = \{k_{t-1}, \ell_t\}$.

Lab 15

Use the law of motion and approximate policy function from homework 14 to generate 1000 artificial time series for an economy where each time series is 250 periods long. Start each simulation off with a starting value for k that is ten percent below the steady state value, and a value of z that is one standard deviation below zero.

Use $\sigma_z^2 = .004$.

For each simulation save the time-series for GDP, consumption and investment. When all 1000 simulations have finished generate a graph for each of these time-series showing the average value over the simulations for each period, and also showing the five and ninety-five percent confidence bands for each series each period.

In addition, calcuate the mean, standard deviation, autocorrelation and correlation with output for each series over each simulation and report the average values and standard deviations for these moments over the 1000 simulations.

Lab 16

Using the same setup as homework 14, generate impulse response functions for GDP, consumption and investment with lags from zero to forty periods.

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