Econ 413R: Computational Economics

Spring Term 2013

Perturbation Methods for DSGE Models

1 Introduction

In this section we will explore in more detail the perturbation methods referenced in section 5.4 of the DSGE chapter. We will only consider a second order approximation of the policy function here, but approximations of yet higher order follow the same basic approach. There are many alterations of the standard perturbation method. Detailed discussions of perturbation methods can be found in chapters 13 – 15 of Judd (1998), as well as in Collard and Juillard (2001), Schmitt-Grohe and Uribe (2004), and Heer and Maussner (2009).

As noted previously, assuming that the policy functions are linear can be extremely useful in solving DSGE models. When second or higher order properties of the characterizing equations of the are important to the question being answered, however, linearization is undesirable. Linearization is essentially a first order Taylor series approximation about the steady state, as discussed in section 6 of the DSGE chapter. Such an approximation results in certainty equivalence, or the phenomenon of unconditional expectations of the endogenous variables being equal to their non-stochastic steady state values. This occurs because in a linear approximation, only the first moments of the shocks enter the linear equations. As these processes are assumed to be mean zero, these moments wash out when expectations are taken. Thus, the distribution of the shocks have no influence on the resultant policy equation solutions.

Applications where this can be troublesome include asset pricing models and welfare analysis. In asset pricing models the riskiness of an asset is directly related to the variance of the underlying shocks. Thus, failing to account for higher order characteristics of the model can invalidate the results. In welfare analysis where the utility functions have high curvature, failing to account for the second moment can similarly produce spurious results.

2 Pertubation Methods in General

To see how perturbation methods work consider the following simple example. Suppose we have a condition on a potentially nonlinear bivariate function:

$$F(x, u) = 0 \tag{2.1}$$

Assume u is an exogenously given variable, and x will be choson to satisfy (2.1). Denote the solution to this condition as $x(u)$ and assume that the value of $x(u_0)$ is known.

Taking the derivative of (2.1) with respect to u gives:

$$F_x\{x(u), u\}x_u(u) + F_u\{x(u), u\} = 0 \tag{2.2}$$

If we evaluate this at $u = u_0$ and solve for the first derivative of $x(u)$, we have:

$$x_u(u_0) = -\frac{F_u\{x(u_0), u_0\}}{F_x\{x(u_0), u_0\}} \tag{2.3}$$

Since $x(u_0)$ is known, as long as $F_x\{x(u_0), u_0\} \neq 0$ we can find the value for the first derivative. The first-order (linear) Taylor-series approximation of $x(u)$ will be:

$$x(u) = x(u_0) + x_u(u_0)(u - u_0) \tag{2.4}$$

To find the second-order terms we differentiate (2.2) again with respect to u.

$$\begin{aligned}
&F_{xx}\{x(u), u\}x_u(u)x_u(u) + F_{xu}\{x(u), u\}x_u(u)\\
&F_x\{x(u), u\}x_{uu}(u)\\
&F_{xu}\{x(u), u\}x_u(u)\\
&F_{uu}\{x(u), u\} = 0
\end{aligned} \tag{2.5}$$

Again evaluating at $u = u_0$ and solving this time for the second derivative of $x(u)$, we have:

$$x_{uu}(u_0) = -\frac{F_{xx}\{x(u_0), u_0\}[x_u(u_o)]^2 + 2F_{xu}\{x(u_0), u_0\}x_u(u_o) + F_{uu}}{F_x\{x(u_0), u_0\}} \tag{2.6}$$

Hence, the second-order (quadratic) Taylor-series approximation of $x(u)$ will be:

$$x(u) = x(u_0) + x_u(u_0)(u - u_0) + \tfrac{1}{2}x_{uu}(u_0)(u - u_0)^2 \tag{2.7}$$

Higher order terms can be obtained by successive differentiation of (2.5), setting $u = u_0$ and solving for the appropriate derivative. This will be a function of the various derivatives of $F(x, u)$ and the lower-order derivatives of $x(u)$ obtained from previous iterations.

3 Pertubation Methods in Dynamic Systems

Recall our system of dynamic equations from the linearization chapter. We can take natural logs or otherwise transfomr the equatuions to get:

$$E_t\{\Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t)\} = 0 \tag{3.1}$$

We have already shown one method for obtaining a linear approximation of the policy function. Our task in this section is to obtain the quadratic terms from a second-order approximation of the same policy function. As with the linearization discussion we will assume that there are no jump variables, though if some of the variables included in the list of endogenous state variables are, in fact, jump variables, this is not a problem.

We must recall that the exogenous state variables evolve according to a linear law of motion given in (3.2).

$$\tilde{Z}_t = N\tilde{Z}_{t-1} + \sigma\Omega\varepsilon_t; \varepsilon_t \sim (0, I_{n_Z}) \tag{3.2}$$

where σ is a scalar, and Ω is a matrix that determines correlations of the elements in ε_t.

We are searching for the quadratic terms in the Taylor-series approximation of the policy function which we will denote $X_t = H(X_{t-1}, Z_t, \sigma)$.

The second-order Taylor-series approximation of row i of H is:

$$
\begin{aligned}
H^i(X_{t-1}, Z_t, \sigma) &= H^i(\bar{X}, \bar{Z}, 0) + \begin{bmatrix} H_X^i & H_Z^i \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \end{bmatrix} \\
&+ \tfrac{1}{2} \begin{bmatrix} \tilde{X}_{t-1}^T & \tilde{Z}_t^T & \sigma \end{bmatrix} \begin{bmatrix} H_{XX}^i & H_{XZ}^i & 0 \\ H_{ZX}^i & H_{ZZ}^i & 0 \\ 0 & 0 & H_{\sigma\sigma}^i \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \sigma \end{bmatrix}
\end{aligned}
\tag{3.3}
$$

The H_X^i and H_X^i terms are the approprite rows of the P and Q matrices in Uhlig's notation.

3.1 Useful Definitions and Notation

We make the following useful definitions.

$$
H \equiv \begin{bmatrix} H^1(X_{t-1}, Z_t, \sigma) \\ \vdots \\ H^{n_X}(X_{t-1}, Z_t, \sigma) \end{bmatrix}
$$

$$
H_X \equiv \begin{bmatrix} H_{X^1}^1 & \dots & H_{X^{n_X}}^1 \\ \vdots & \ddots & \vdots \\ H_{X^1}^{n_X} & \dots & H_{X^{n_X}}^{n_X} \end{bmatrix}
$$

$$
H_Z \equiv \begin{bmatrix} H_{Z^1}^1 & \dots & H_{Z^{n_Z}}^1 \\ \vdots & \ddots & \vdots \\ H_{Z^1}^{n_X} & \dots & H_{Z^{n_Z}}^{n_X} \end{bmatrix}
$$

Note that H is an $n_X \times 1$ vector, H_X is an $n_X \times n_X$ matrix and H_Z is an

$n_X \times n_Z$ matrix.

We also define a $n_X \times n_X$ matrix S_X and a $n_X \times n_Z$ matrix R_Z.

$$S_X = H_X H_X$$

$$R_Z = H_X H_Z + H_Z N$$

We define three three-dimensional tensors, H_{XX}, H_{ZZ} and H_{XZ}, which are $n_X \times n_X \times n_X$, $n_X \times n_Z \times n_Z$ and $n_X \times n_X \times n_Z$. respectively. $H_{\sigma\sigma}$ is an $n_x \times 1$ vector.

In (3.3) an i superscript on any of the tensors, H_{XX}, H_{ZZ} and H_{XZ}, denotes the i^{th} slice in the third dimension, which indexes the equation in (3.1). For $H_{\sigma\sigma}$ an i superscript denotes the i^{th} element in the vector. For H_X, H_Z, S_X and R_Z it denotes the i^{th} column of the matrix which indexes the variable being differentiated.

For derivative matrices of the Γ function we adopt the notation that $\Gamma^i_{[X_{t+1}, X_t, x^j_{t-1}]}$ denotes a column vector of first derivatives of the i^{th} equation with respect to the vector X_{t+1}, the vector X_t, and the scalar x^j_{t-1}, which is the j^{th} element in the vector X_{t-1}. $\Gamma^i_{[\dots][\dots]}$ will denote a matrix of second derivatives with the first sqare bracket indicating the rows and the second indicating the columns of the variables.

3.2 Linear Terms

Differentiating the i^{th} equation in (3.1) with respect to x_{t-1}^j and evaluating at $(\bar{X}, \bar{Z}, \sigma)$ gives:

$$\begin{bmatrix} {S_X^j}^T & {H_X^j}^T & 1 \end{bmatrix} \Gamma^i_{[X_{t+1}, X_t, x_{t-1}^j]} = 0 \tag{3.4}$$

Note that (3.4) defines a set of ${n_X}^2$ equations are the conditions that implicitly define the ${n_X}^2$ first-order terms in H_X. (P in Uhlig's notation.)

We can also differentiate (3.1) with respect to z_t^j and evaluate at $(\bar{X}, \bar{Z}, \sigma)$ to get:

$$\begin{bmatrix} {R_Z^j}^T & {H_Z^j}^T & {N^i}^T & 1 \end{bmatrix} \Gamma^i_{[X_{t+1}, X_t, Z_{t+1}, z_t^j]} = 0 \tag{3.5}$$

where N^i is i^{th} column of the N matrix in (3.2). This set of $x_X n_Z$ equations are the conditions that implicitly define the $x_X n_Z$ first-order terms in H_Z. (Q in Uhlig's notation.)

3.3 Quadratic Terms

In this subsection we will use the perturbation results from section 2 above.

Differentiating (3.4) with respect to x_{t-1}^k gives:

$$\begin{aligned} &\begin{bmatrix} {S_X^j}^T & {H_X^j}^T & 1 \end{bmatrix} \Gamma^i_{[X_{t+1}, X_t, x_{t-1}^j][X_{t+1}, X_t, x_{t-1}^k]} \begin{bmatrix} S_X^k \\ H_X^k \\ 1 \end{bmatrix} \\ &+ \left[\Gamma^i_{[X_{t+1}]} H_X + \Gamma^i_{[X_t]} \right] H_{XX}^{jk} = 0 \end{aligned}$$

where H_{XX}^{jk} is a $n_X \times 1$ vector from the three-dimensional tensor H_{XX}

This can be rewritten as:

$$
\begin{aligned}
&\left[\Gamma^i_{[X_{t+1}]} H_X + \Gamma^i_{[X_t]}\right] H^{jk}_{XX} \\
&= -\begin{bmatrix} {S^j_X}^T & {H^j_X}^T & 1 \end{bmatrix} \Gamma^i_{[X_{t+1},X_t,x^j_{t-1}][X_{t+1},X_t,x^k_{t-1}]} \begin{bmatrix} S^k_X \\ H^k_X \\ 1 \end{bmatrix}
\end{aligned}
\tag{3.6}
$$

This is a system of ${n_X}^3$ equations and unknowns that implicitly defines the elements of the three-dimensional tensor H_{XX}. By vectorizing the elements in H_{XX} this system can be written in the form $AH = q$ with $H = vec\{H_{XX}\}$.

Similarly, differentiating (3.4) with respect to z^k_t reveals that the elements of H_{XZ} solve:

$$
\begin{aligned}
&\left[\Gamma^i_{[X_{t+1}]} H_X + \Gamma^i_{[X_t]}\right] H^{jk}_{XZ} \\
&= -\begin{bmatrix} {S^j_X}^T & {H^j_X}^T & 1 \end{bmatrix} \Gamma^i_{[X_{t+1},X_t,x^j_{t-1}][X_{t+1},X_t,Z_{t+1},z^k_t]} \begin{bmatrix} R^k_X \\ H^k_X \\ N^i \\ 1 \end{bmatrix}
\end{aligned}
\tag{3.7}
$$

This is a system of ${n_X}^2 n_Z$ equations and unknowns that implicitly defines the elements of H_{XZ}, which can also be written in the form $AH = q$.

Differentiating (3.5) with respect to z_t^k shows that elements of H_{ZZ} solve:

$$
\begin{aligned}
&\left[\Gamma^i_{[X_{t+1}]} H_Z + \Gamma^i_{[X_t]}\right] H_{ZZ}^{jk} \\
&= - \begin{bmatrix} {R_X^k}^T & {H_X^k}^T & {N^i}^T & 1 \end{bmatrix} \Gamma^i_{[X_{t+1}, X_t, x^j_{t-1}][X_{t+1}, X_t, Z_{t+1}, z^k_t]} \begin{bmatrix} R_X^k \\ H_X^k \\ N^i \\ 1 \end{bmatrix}
\end{aligned}
\tag{3.8}
$$

This is a system of $n_X {n_Z}^2$ equations and unknowns that implicitly defines the elements of H_{ZZ}

The elements of $H_{\sigma\sigma}$ solve:

$$
H_{\sigma\sigma}^T \Gamma^i_{[X_{t+1}]} = - \begin{bmatrix} \Delta_1 & \dots & \Delta_{n_Z} \end{bmatrix} \Gamma^i_{[X_{t+1}, Z_{t+1}][X_{t+1}, Z_{t+1}]} \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_{n_Z} \end{bmatrix} \tag{3.9}
$$

where $\Delta_i \equiv \sum_{s=1}^{n_Z} \omega_{is} \varepsilon^s_{t+1}$ and ω_{rc} is the r^{th} row and c^{th} column of the Ω matrix.

This is a system of n_X equations and unknowns that implicitly defines the elements of $H_{\sigma\sigma}$.

4 Applying Perturbation Methods to the Brock and Mirman Model

In the model from Brock and Mirman (1972) we will have $H_X = P$ and $H_Z = Q$ from the linearization exercise. Equations (3.4) and (3.5) reduce to:

$$\begin{aligned} P^2F + PG + H &= 0 \\ (PQ + QN)F + QG + NL + M &= 0 \end{aligned} \tag{4.1}$$

Which are the same conditions as we got from linearization earlier.

Equations (3.6) – (3.8) reduce to:

$$H_{XX} = -\frac{\begin{bmatrix} P^2 & P & 1 \end{bmatrix} \Gamma^i_{[X_{t+1}, X_t, x^j_{t-1}][X_{t+1}, X_t, x^k_{t-1}]} \begin{bmatrix} P^2 \\ P \\ 1 \end{bmatrix}}{FP + G} \tag{4.2}$$

or

$$H_{XX} = -\frac{\begin{bmatrix} P^4\Gamma_{[X_{t+1}][X_{t+1}]} + P^2\Gamma_{[X_t][X_t]} + \Gamma_{[X_{t-1}][X_{t-1}]} \\ +2P^3\Gamma_{[X_{t+1}][X_t]} + 2P\Gamma_{[X_t][X_{t-1}]} + 2P^2\Gamma_{[X_{t+1}][X_{t-1}]} \end{bmatrix}}{FP + G} \tag{4.3}$$

$$
H_{XZ} = -\frac{\begin{bmatrix} P^2 & P & 1 \end{bmatrix} \Gamma^i_{[X_{t+1},X_t,x^j_{t-1}][X_{t+1},X_t,Z_{t+1},z^k_t]} \begin{bmatrix} PQ+QN \\ Q \\ N \\ 1 \end{bmatrix}}{FP+G} \tag{4.4}
$$

$$
H_{ZZ} = -\frac{\begin{bmatrix} PQ+QN & Q & N & 1 \end{bmatrix} \Gamma^i_{[X_{t+1},X_t,x^j_{t-1}][X_{t+1},X_t,Z_{t+1},z^k_t]} \begin{bmatrix} PQ+QN \\ Q \\ N \\ 1 \end{bmatrix}}{FP+G} \tag{4.5}
$$

5 Applying Perturbation Methods to Other Models

As we can see from the previous example, it is critically important to keep accurate track of the elements in the various vectors, matrices and tensors. To implement perturbation methods with other models there are preprogammed software packages that make the accounting for variables easy. The most commonly used is a package called Dynare which runs in conjunction with MATLAB or Octave. There is also a stand-alone version and a version that runs with Python, which is called Dolo. Dynare will implement linear, quadratic and cubic approximations of policy functions. We will spend a full lecture exploring some of its capabilities.

Exercises

Homework 1

For the function $F(k', k) = (k^{.35} + .9k - k')^{-2.5} - .95(k'^{.35} + .9k')^{-2.5} = 0$, use perturbation methods to find the cubic approximation of $k' = f(k)$ about the point $k = 0.1$. In this case, $k' = f(0.1) = 0.069986$.

Homework 2a

For the Brock and Mirman model with the default parameter values find the scalar values of H_X, H_X, H_{XX}, H_{XZ}, H_{ZZ} and $H_{\sigma\sigma}$.

Plot the three-dimensional surface plot for the policy function $K' = H(K, z)$. Compare this with the closed form solution from the notes and the two approximations from the previous homework set (numbers 7a and 8a).

Homework 2b

Repeat the above exercise using $k \equiv \ln K$ in place of K as the endogenous state variable.

Lab 3a

Using Dynare replicate the results from the previous homework set, problem 14. Be sure to specify a first-order approximation. Report the linear coefficents in the policy function. Then replicate the moments and IRFs from problems 15 and 16.

Lab 3b

Repeat the above exercise using a second-order approximation of the policy function. Report all linear and quadratic coefficients. Comment on any differences.

Lab 3c

Repeat problem 3a using a third-order approximation of the policy function. Report all linear, quadratic and cubic coefficients. Comment on any differences.

References

Brock, W. A., and L. J. Mirman (1972): "Optimal economic growth and uncertainty: The discounted case," *Journal of Economic Theory*, 4(3), 479–513.

Collard, F., and M. Juillard (2001): "A Higher-Order Taylor Expansion Approach to Simulation of Stochastic Forward-Looking Models with an Application to a Nonlinear Phillips Curve Model," *Computational Economics*, 17(2-3), 125–39.

Heer, B., and A. Maussner (2009): *Dyanamic General Equilibrium Mod-Model: Computational Methods and Applications*chap. Perturbation Methods, pp. 75 – 174. Springer.

Judd, K. L. (1998): *Numereical Methods in Economics*. MIT Press.

Schmitt-Grohe, S., and M. Uribe (2004): "Solving Dynamic General Equilibrium Modes Using a Second-Order Approximation to the Policy Function," *Journal of Economic Dynamics and Control*, 28(10724), 755–775.