Econ 413R: Computational Economics

Spring Term 2013

Structural Vector Autoregressions

Macroeconomists are intersted in estimating and drawing inferences from "simultaneous equations models." These models are intended to characterize the *structure* of the economy.

1 Four Representations

We will consider four different representations of a system of dynamic equations.

1.1 Structural VAR

$$A_0 y_t = A_1 y_{t-1} + \ldots + A_p y_{t-p} + \varepsilon_t, \quad t = -p+1, \ldots, 0, 1, \ldots, T \tag{1.1}$$

where we assume that $Var(\varepsilon_t) = E(\varepsilon_t \varepsilon_t^{'}) \equiv \Omega$, a diagonal matrix.

Note that $y_t,\ t = -p+1, \ldots, 0, 1, \ldots, T$ are $K \times 1$ vectors of observations on economic variables, $\varepsilon_t,\ t = -p+1, \ldots, 0, 1, \ldots, T$ are $K \times 1$ vectors of structural disturbances (shocks), which have economic interpretation, $A_i,\ i = 1, \ldots, p$ are $K \times K$ coefficient matricies. The fact that A_0 is not a diagonal matrix makes this a system of simultaneous equations. Ω is a diagonal covariance matrix which implies that the structural disturbances are uncorrelated.

Our problem is that the structural VAR is not *identified* (uniquely estimable) without sufficient restrictions. Sims (1980) was very critical of the tradition approach to **identification** which imposed *ad hoc* zero restrictions on elements of A_0 *and* $A_1, \ldots, A_p$ without theoretical justification. He argued that it may be more credible to impose identifying restrictions only on A_0 which reflects responses of variables to each other within a time period. The credibility of such restrictions depends on length of the period (frequency) of observations and the kind of variable in question.

We may use the lag operator to rewrite (1.1) as

$$A(L)y_t = \varepsilon_t, \quad t = -p+1, \ldots, 0, 1, \ldots, T \tag{1.2}$$

where $A(L)$ is the following matrix of polynomials of degree p in the lag operator L:

$$A(L) = [A_0 - A_1 L - \ldots - A_p L^p]$$

1.2 Structural MA

Solve (2.1) for y_t to obtain the structural moving average representation:

$$y_t = A(L)^{-1}\varepsilon_t \equiv [C_0 + C_1 L + C_2 L^2 + \ldots]\varepsilon_t = C(L)\varepsilon_t \tag{1.3}$$

Note that the elements of the infinite order matrix polynomial $C(L)$ give the K^2 **impulse response functions (IRFs)**. $C_{ij,h}$ is the response of variable i in h periods to a one unit movement in the j^{th} structrual shock today, period 0. The IRFs are of particular interst to macroeconomists. As the structural VAR representation is not identified, neither is the structual MA representation.

1.3 Reduced Form VAR

We can obtain the reduced form VAR representation of this system by premultiplying (1.1) by A_0^{-1}.

$$y_t = A_0^{-1}A_1 y_{t-1} + \ldots + A_0^{-1}A_p y_{t-p} + A_0^{-1}\varepsilon_t = B_1 y_{t-1} + \ldots + B_p y_{t-p} + u_t \quad (1.4)$$

This representation is sometimes referred to simply as a VAR model. The K equations of the VAR model can be estimated using ordinary least squares (OLS). The reduced form errors (innovations), $u_t = A_0^{-1}\varepsilon_t$, are (as yet) unknown nonlinear functions of the structural shocks and do not have any direct economic interpretation. The variance of u_t is given by

$$Var(u_t) = E(u_t u_t') = (A_0^{-1})\Omega(A_0^{-1})' \equiv \Sigma$$

If we assume $\Omega = I$ (a **normalization**), then $\Sigma = (A_0^{-1})(A_0^{-1})'$. Equivalently we could normalize by assuming the diagonal elements of A_0 are unity.

An alternative way of writing (1.4) is to once again use the lag operator.

$$B(L)y_t = u_t \quad (1.5)$$

where $B(L)$ is the following matrix polynomial of degree p in the lag operator,

L:

$$B(L) = [I - B_1 L - \ldots - B_p L^p]$$

1.4 Reduced Form MA

We solve (1.5) for y_t to obtain the final representation of our system, the reduced form moving average representation.

$$y_t = B(L)^{-1} u_t \equiv [D_0 + D_1 L + D_2 L^2 + \ldots] u_t = D(L) u_t \tag{1.6}$$

where $D(L)$ is an infinite order matrix polynomial.

Note that the following recursion holds:

$$\begin{aligned} D_0 &= I \\ D_j &= \sum_{i=1}^{p} B_i D_{j-i}, \quad j = 1, 2, \ldots \\ D_j &= 0, \quad j < 0 \end{aligned}$$

Equation (1.6) is an impulse response function, but the reduced form errors, u_t, have no economic interpretation except as one-step-ahead forecast errors. By the definition of $B(L)$ and the fact that $u_t = A_0^{-1} \varepsilon_t$, we can write the reduced form MA representation in terms of the structural parameters.

$$y_t = D(L) u_t = [A_0^{-1} A(L)]^{-1} A_0^{-1} \varepsilon_t \tag{1.7}$$

2 Identification

How can we recover estimates of the structural parameters in (1.1) and (1.3) from knowledge of the reduced form parameters in equations (1.4) and (1.6)? In particular, we would like to estimate the impulse response functions given in (1.3). There are two ways to consider identifying (1.3). The first requires knowledge of A_0 to go from the observable (estimable) reduced form VAR (1.4) to the SVAR (1.1). The second requires knowledge of A_0 to go from the observable reduced form MA representation (1.6) to the structural MA representation (1.3).

2.1 From (1.4) to (1.1)

Note that $A_0 B(L) = A(L)$ and $A_0 u_t = \varepsilon_t$. Thus we can obtain (1.1) by pre-multiplying (1.4) by A_0. We can then invert (1.1) to obtain the desired impulse response functions, (1.3). Thus a knowledge of A_0 is sufficient to identify the IRF.

2.2 From (1.6) to (1.3)

Recall (1.3) and (1.6) and the fact that $D_0 = I$.

$$y_t = A(L)^{-1}\varepsilon_t = [C_0 + C_1 L + C_2 L^2 + \ldots]\varepsilon_t = C(L)\varepsilon_t \tag{2.1}$$

$$y_t = B(L)^{-1}u_t = [I + D_1 L + D_2 L^2 + \ldots]u_t = D(L)u_t \tag{2.2}$$

Equating terms we have $C_0\varepsilon_t = u_t$ for all t and $C_s\varepsilon_{t-s} = D_s u_{t-s} =$

$D_s C_0 \varepsilon_{t-s}$ for all t and for $s = 0, 1, 2, \ldots$ which implies $C_s = D_s C_0$. Thus knowledge of C_0 is sufficient to identify the IRF. But $C_0 \varepsilon_t = u_t = A_0^{-1} \varepsilon_t$ implies that $C_0 = A_0^{-1}$. So, as in the previous case, we see that knowledge of A_0 is sufficient to identify the desired IRF.

2.3 Identifiying A_0

Recall that the reduced form disturbance covariance matrix is $Var(u_t) = \Sigma = (A_0^{-1})(A_0^{-1})'$ if we normalize, by assuming $\Omega = I$. Observable Σ is symmetric and therefore only has $\frac{K(K+1)}{2}$ unique elements. A_0 has K^2 unique elements. If we can find enough restrictions on A_0 such that the remaining parameters in A_0 are uniquely determined by Σ then then model is exactly identified. The necessary condition for this identification is that we impose $K^2 - \frac{K(K+1)}{2} = \frac{K(K-1)}{2}$ restrictions. These restrictions can come from a number of sources including economic theory, information lags, and physical constraints. Once these restrictions are imposed we can obtain A_0 using $\Sigma = (A_0^{-1})(A_0^{-1})'$ with a nonlinear equation solver. After obtaining A_0, we can recover $\varepsilon_t = A_0 u_t$ and construct the impulse responses by $C_s = D_s A_0^{-1}, \quad s = 0, 1, 2, \ldots$ There are two leading approaches to obtaining these addtional $\frac{K(K-1)}{2}$ identifying restrictions: short-run restrictions and long-run restrictions.

2.3.1 Short-Run Restrictions

Impose $\frac{K(K-1)}{2}$ restrictions directly on A_0. These can be zero restrictions, but need not be. Assuming that the $(i, j)^{th}$ element of A_0 is zero implies that the i^{th} variable does not respond contemporaneously to the j^{th} structural shock. Such an assumption may be credible depending on the sampling frequency and on the nature of the variables. For example, it is more reasonable to assume that a monetary policy shock has no effect on the unemployment rate within a month than within a year. Similarly, it is more reasonable to assume that a monetary policy shock has no contemporaneous effect on the unemployment rate than no effect on the market interest rate.

The most frequently used approach is to assume that A_0 is a triangular matrix with the $\frac{K(K-1)}{2}$ terms above the diagonal equal to zero. This structure imposes a *Cholesky decomposition* which implies a recursive structure on the model. This is a very special structure that is rarely justified by economic theory.

2.3.2 Long-Run Restrictions

Note that the long run (cumulative) effects of shocks on the variables of the model are given by adding up the short-run effects: $C(1) = [C_0 + C_1 + C_2 + \ldots]$ where we have substituted $L = 1$. Long-run restrictions are imposed on $C(1)$ which imply restrictions on A_0. For example, we may assume that the $(i, j)^{th}$ element of $C(1)$ is zero. This implies that a change in the j^{th} shock has no long-run effect on the i^{th} variable. Recall from (1.3) and (1.6) that $y_t = C(L)\varepsilon_t = D(L)u_t$. This implies the equivalence of the corresponding long-run covariance matrices

$$D(1)\Sigma D(1)' = C(1)\Omega C(1)' = C(1)C(1)' \tag{2.3}$$

The left-hand-side of (2.3) is observable with $\frac{K(K+1)}{2}$ unique elements. The right-hand-side has K^2 unobserved elements so we need (at least) $\frac{K(K-1)}{2}$

restrictions on $C(1)$. These can come from assumptions that certain shocks have no long-run effects on certain variables which imply specific zero elements in $C(1)$. We can easily see that these long-run restrictions are, in fact, restrictions on A_0 when we pre-multiply both sides of (2.3) by $D(1)^{-1}$ and post-multiply by $[D(1)']^{-1}$. Notice that $D(L)^{-1}C(L) = B(L)A(L)^{-1} = A_0^{-1}$.

3 Examples

We consider two simple examples that illustrate how to obtain impulse response functions under these two identification strategies. Both are based on a model investigated by Blanchard and Quah (1989). They only consider a long-run restriction, but we will use their model to think about a short-run restriction as well.

3.1 Short-Run Restriction

Blanchard and Quah (BQ) consider a two variable/equation model ($K = 2$) with

$$y_t = \begin{pmatrix} \Delta RGDP_t \\ U_t \end{pmatrix}$$

where U is the unemployment rate as a percent and $RGDP$ is measured as the natrual logarithm of real GDP. We only need one restriction since $\frac{K(K-1)}{2} = 1$. We want to interpret the structural shocks as aggregate supply and aggregate demand shocks and do so by assuming that an aggregate demand shock will have no contemporaneous effect on the unemployment rate. This reflects the widely-held belief that unemployment reacts with a lag to aggregate demand shocks. We can impose this restriction in the representation of the reduced form VAR where we denote the elements of A_0^{-1} as a_{ij}.

$$\begin{pmatrix} \Delta RGDP_t \\ U_t \end{pmatrix} = B_1 y_{t-1} + \ldots + B_p y_{t-p} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{pmatrix} \varepsilon_t^{AS} \\ \varepsilon_t^{AD} \end{pmatrix}$$

(This would be a standard triangular matrix if we reordered the elements of y_t.) We can estimate the IRF using the following steps:

i. Obtain $B_s,\ s = 1, 2, \ldots, p,\ u_t,\ t = 1, \ldots, T$, and Σ by estimating the reduced form VAR using OLS (including a constant term or measuring the variables as deviations from their means).

ii. Using an equation solver, recover the restricted A_0^{-1} from

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & 0 \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} \\ a_{11}a_{21} & a_{21}^2 \end{bmatrix}$$

iii. Obtain $D_s,\ s = 0, 1, 2, \ldots$ using the recursion given in Section 1.4.

iv. Recover $\varepsilon_t = A_0 u_t$ and construct the four structural impulse response functions from

$$C_s = D_s A_0^{-1}, \quad s = 0, 1, 2, \ldots$$

Recalling that $D_0 = I$, we can then plot the desired IRFs for $s = 1, 2, \ldots, H$ where H is a chosen horizon (e.g. 40 quarters).

Note: We usually want to plot the IRFs for the *level* of RGDP, not its changes, so for each horizon we must cumulate the impulse responses for changes up to that horizon; i.e. the impulse response for the level of RGDP in response to an aggregate supply shock at horizon h is computed as $\sum_{s=0}^{h} C_{11,s}$.

3.2 Long-Run Restriction

Consider the same pre-restriction two-variable model as in the previous example. As before, we need one addtional restriction to identify the model. In this example we follow Blanchard and Quah (1989) by assuming that the aggregate demand shock has no long-run effect on the *level* of RGDP as implied by most macroeconomic theories. Since the variable in y_t is RGDP *growth*, this is a restriction on the cumulative effect on output growth rates of the aggregate demand shock which is a zero restriction on the (1,2) element of $C(1)$.

We can estimate the IRF using the following steps:

i. Obtain the unknown elements of $C(1)$ as follows:

- Obtain $B_s,\ s = 1, 2, \ldots, p$, $u_t,\ t = 1, \ldots, T$, and Σ by estimating the reduced form VAR using OLS (including a constant term or measuring the variables as deviations from their means).
- Compute $D(1) = B(1)^{-1}$ where $B(1) = I - B_1 - \ldots - B_p$.
- Use an equation solver to obtain $C(1)$ from (2.1).

$$D(1)\Sigma D(1)' = C(1)C(1)' = \begin{bmatrix} C_{11}(1) & 0 \\ C_{21}(1) & C_{22}(1) \end{bmatrix} \begin{bmatrix} C_{11}(1) & C_{21}(1) \\ 0 & C_{22}(1) \end{bmatrix}$$

ii. Recover A_0^{-1} from $D(1)^{-1}C(1) = A_0^{-1}$

iii. Obtain $D_s,\ s = 0, 1, 2, \ldots$ using the recursion given in Section 1.4.

iv. Recover $\varepsilon_t = A_0 u_t$ and construct the four structural impulse response functions from

$$C_s = D_s A_0^{-1}, \quad s = 0, 1, 2, \ldots$$

Exercises

Homework 1

SVAR with a short-run restriction

Construct the four impulse response functions (IRFs) for the two-variable model of Example 1 in section 3.1 with

$$y_t = \begin{pmatrix} U_t \\ \Delta RGDP_t \end{pmatrix}$$
$$\varepsilon_t = \begin{pmatrix} \Delta\varepsilon_t^{AS} \\ \varepsilon_t^{AD} \end{pmatrix}$$

where $\Delta RGDP_t$ is the difference of the log of real GDP (i.e, the rate of growth of real GDP) and U_t is the unemployment rate as a percent. Note that I have reordered the variables in order to get a standard triangular A_0^{-1} matrix. [Data can be downloaded from the FRED site available at the website of the Federal Reserve Bank of St. Louis.] For purposes of comparison, use quarterly data from 1948:1-1987:4 and make no other adjustments to the data. Use the assumption that AD shocks have no contemporaneous effects on the unemployment rate. Estimate your VAR model using eight lags (i.e., p = 8) and choose a horizon of 40 quarters for your impulse response functions (i.e., $H = 40$). Recall that we are interested in the IRF for the *level* of RDGP (which is the natural log of real GDP)

Homework 2

SVAR with a long-run restriction Construct the four IRFs for the two-variable model of Example 2 in section 3.2 with

$$y_t = \begin{pmatrix} \Delta RGDP_t \\ U_t \end{pmatrix}$$
$$\varepsilon_t = \begin{pmatrix} \Delta \varepsilon_t^{AS} \\ \varepsilon_t^{AD} \end{pmatrix}$$

where the variables are measured as in Homework 1. Once again, use quarterly data from 1948:1-1987:4 and make no other adjustments to the data. In this problem, however, use that assumption that AD shocks have no long-run effect on the (log) level of real GDP. As in the previous problem, estimate your VAR model using eight lags (i.e., p = 8) and choose a horizon of 40 quarters for your impulse response functions (i.e., H= 40). Once again, recall that we are interested in the IRF for the *level* of RDGP (which is the natural log of real GDP)

References

Blanchard, O. J., and D. Quah (1989): "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review*, 79(4), 655–73.

Sims, C. A. (1980): "Macroeconomics and Reality," *Econometrica*, 48(1), 1–48.